Blowing-up of locally monomially foliated space

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Abstract

In this paper, we prove that the blowing-up preserve the local monomiality of foliated space.

1 Locally monomial foliations

Let M be an analytic manifold of dimension n and $D \subset M$ be a divisor with normal crossings. We denote respectively by $\mathcal{O}_M$ and $\Theta_M[\log D]$ the sheaf of holomorphic functions and the sheaf of vector fields on M which are tangent to D.

A singular foliation on (M, D) is coherent subsheaf $\mathcal{F}$ of $\Theta_M[\log D]$ which is reduced and integrable (see [1] and [2]). The dimension (or the rank) of $\mathcal{F}$ is given by

$$s = \max_{p \in M} \dim \mathcal{F}(p)$$

where $\mathcal{F}(p) \subset T_pM$ denote the vector subspace generated by the evaluation of $\mathcal{F}$ at p.

Let $\mathbb{F}$ be a field (we usually take $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$). We shall say that $\mathcal{F}$ is $\mathbb{F}$-*locally monomial* if for each point $p \in M$ there exists

1. a local system of coordinates $x = (x_1, \ldots, x_n)$ at p
2. an s-dimensional vector subspace $V \subset \mathbb{F}^n$

such that D is locally given by

$$D_p = \{x_i = 0 : i \in I\}, \quad \text{for some} \quad I \subset \{1, \ldots, n\}$$

and $\mathcal{F}_p$ is the $\mathcal{O}_{M,p}$-module generated by the abelian Lie algebra

$$\mathcal{L}(V) = \{\sum_{i=1}^{n} a_i x_i \frac{\partial}{\partial x_i} : a \in V\} \bigoplus_{i\bar{I}: e_i \in M} \mathbb{F}\frac{\partial}{\partial x_i}$$

where $\bar{I} = \{1, \ldots, n\} \setminus I$. We shall say that the triple $(M, D, \mathcal{F})$ is locally monomially foliated space and that (x, I, V) is a local presentation at p.

Lemma 1.1. *Let (x, I, V) be a local presentation for $(M, D, \mathcal{F})$ at a point p. Then, for each vector $m \in V^\perp$, the (possibly multivalued) function $f(x) = x^m$ is a first integral of $\mathcal{F}$.*

2 Blowing-up

Let $Y \subset M$ be a smooth submanifold of codimension r. We shall say that Y has *normal crossings* with $(M, D, \mathcal{F})$ if for each point $p \in Y$ there exists a local presentation (x, I, V) at p such that Y is given by

$$Y = \{x_1 = x_2 = \ldots = x_r = 0\}.$$

Proposition 1.2. *Let $\Phi : \widetilde{M} \to M$ be the blowing-up with a center Y which has normal crossings with $(M, D, \mathcal{F})$. Let*

$$\widetilde{D} = \Phi^{-1}(D) \quad \textit{and} \quad \widetilde{\mathcal{F}} \subset \Theta_{\widetilde{M}}[\log \widetilde{D}]$$

denote the total transform of D and the strict transform of $\mathcal{F}$ respectively. Then, the triple $(\widetilde{M}, \widetilde{D}, \widetilde{\mathcal{F}})$ is a locally monomially foliated space.

The proof is based on the following result on linear algebra.

3 Some linear algebra

Let $\mathbb{F}$ be a field and let $V \subset \mathbb{F}^n$ be a vector subspace of dimension s. Let us fix a disjoint partition of indices $\{1, \ldots, n\} = I_1 \sqcup I_2$ write $\mathbb{F}^n = \mathbb{F}^{I_1} \oplus \mathbb{F}^{I_2}$ and let $\pi_I : \mathbb{F}^n \to \mathbb{F}^I$ denote the projection in the corresponding subspace $\mathbb{F}^I$ generated by $\{e_i : i \in I\}$.

Lemma 3.1. *There exists a basis for V such that for each vector v in this basis, either*

$$v = \pi_{I_2}(v) \qquad \textit{or} \qquad v = \pi_{I_2}(v) + e_i$$

for some $i \in I_1$

Proof. Up to a permutation of coordinates, we can suppose that $I_1 = \{1, \ldots, n_1\}$ and $I_1 = \{n_1 + 1, \ldots, n\}$ (with the convention that $n_1 = 0$ if $I_1 = \emptyset$)

Let $M = [m_1, \ldots, m_s]$ be a $s \times n$ matrix whose rows are an arbitrary basis of V. By a finite number of elementary row operations and permutations of columns (which leave invariant the subsets I_1 and I_2), we can suppose that the matrix M has the form

$$M = \left(\begin{array}{c|c||c|c} Id_{k_1 \times k_1} & A_{k_1 \times l_1} & 0_{k_1 \times k_2} & B_{k_1 \times l_2} \\ \hline 0_{k_2 \times k_1} & 0_{k_2 \times l_1} & Id_{k_2 \times k_2} & C_{k_2 \times l_2} \end{array} \right)$$

where $k_i + l_i = |I_i|$ for $i = 1, 2$, $k_1 + k_2 = s$ and $Id_{k,l}$ and $0_{k,l}$ denote the $k \times l$ identity and zero matrix respectively.

Now, it suffices to define $v_i = e_i + \sum_{n_1+k_2+1 \leq j} m_{i,j} e_j$, for $i = 1, \ldots, k_1$ and $v_i = e_{i+n_1} + \sum_{n_1+k_2+1 \leq j} m_{i,j} e_j$ for $i = k_1 + 1, \ldots, s$. $\square$

References

[1] Paul Baum and Raoul Bott, *Singularities of holomorphic foliations.* J. Differential Geometry, 7: 279-342, 1972.

[2] Yoshiki Mitera and Junya Yoshizaki. *The local analytical triviality of a complex analytic singular foliation.* Hokkaido Math. J., *33(2):275-297, 2004*

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